The Analysis of Concrete Cross Sections under Arbitrary Loading  
using a Consistent Theory

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Summary

Arbitrary loading involving not only bending but shear and torsion acting on a concrete cross section cannot be analysed with the existing codes using a consistent theory. This applies both to the serviceability and to the ultimate limit state. For this reason, a theory is introduced here which combines the benefits of the classical bending theory (TB) with those of the theory of plane stress. This extension of the classical bending theory (ETB) ensures that all required effects can be analysed with the necessary accuracy, particularly in cases of arbitrary loading including shear and torsion. This will be outlined in the paper. Using this theory, a cross section of any shape can be handled and various materials, including reinforced and prestressed concrete, can be considered.

Keywords: arbitrary loading; concrete cross section; serviceability limit state; prestressing; ultimate limit state.

1. Introduction

The analysis of pure bending problems in reinforced and prestressed concrete structures does not involve any difficulties. For cross sections of any shape an iterative procedure, which can be easily performed and which is based on the classical bending theory, is the evident solution. This theory is valid for all states; the serviceability limit state (SLS) as well as the ultimate limit state (ULS). Furthermore, based on this theory, the fatigue behaviour can be analysed and also cracking can be calculated under the assumption of pure bending.

Shear and torsion, however, cannot be handled in the same easy way. Models based on the equilibrium method are therefore introduced in the concrete codes. These models are mechanically correct only for the ultimate limit state as they are based on the theory of plasticity. For this reason, the serviceability limit state can only be analysed using semi-empirical models of varying accuracy. The differences between the various concrete codes demonstrate that for such cases a mechanically
consistent model – one that is valid throughout all limit states – does not exist. The current corrections and additional restrictions in the national documents of the Eurocode 2, concerning shear and torsion, demonstrate the shortcomings of the existing theories.

According to these codes (see e.g. Eurocode No. 2 [1]), a design analysis for pure bending with longitudinal force has to be performed. This has to be followed by an individual analysis for pure shear, pure torsion and for the combination of shear and torsion. Finally, a simplified analysis (check of the critical principal stress) for the combination of bending and torsion needs to be carried out. In cases of arbitrary loading all the above analyses at ULS have to be performed. For a combination of loading including shear and torsion or bending and torsion, the given formulae in the codes are empirical. A direct investigation of an arbitrary loading consisting of bending ($M_y$, $M_z$), longitudinal force ($N$), shear ($V_y$, $V_z$) and torsion ($T_x$) is not possible using existing methods. Furthermore, shear and torsion at SLS are not taken into account for the limitation of stresses and for the control of cracking.

The shortcomings of the above mentioned limit state analysis can be avoided by combining the well known classical bending theory (TB) with elements of the theory of plane stress. Using this extended bending theory (ETB) any arbitrary load cases including shear and torsion ($N$, $V_y$, $V_z$, $T_x$, $M_y$, $M_z$) can be analysed with the necessary accuracy. The analysis is valid throughout all stages including SLS and ULS. Because any load case can be analysed at once, further analysis of the individual actions becomes superfluous. As a result, the analysis of the limit states is not only simplified considerably but is of a higher accuracy as well.

In Retzepis, Hartung [2], Hartung, Krebs [3], Krebs, Schnell, Hartung [4] the theory has already been explained. For this reason only the basics of the theory are presented here. The emphasis of this article lies on the extensive verification tests that have been carried out to check the theory and prove its effectiveness. Some recent practical applications of the theory demonstrate its advantages over the existing methods of analysis for beam structures.

2. Theory

As mentioned before, the combination of the classical bending theory on the cross section (x-face) with the theory of plane stress applied on each element of the cross section (on the local l-q-plane, see Fig. 1) ensures that all required effects can be calculated with the necessary accuracy for both serviceability and ultimate limit state, see Retzepis, Hartung [2].

[Fig. 1 Force and strain notation at a cross section]

In addition to the forces and strains used in the classical bending theory, additional forces, strain and distortional quantities need to be introduced for the extended bending theory. Fig. 1 shows an
arbitrary cross section with the necessary forces and strain notation that are used. The cross section is idealized by its middle line. In the general case, 6 internal global forces \((N, V_y, V_z, T_x, M_y, M_z)\) – which have been evaluated using e.g. a classical structural analysis of beams and frames – and 5 external global forces \((p_x, p_y, p_z, m_x, p_r)\) act on the cross section. The corresponding 11 global strains and distortion values, including their first and second derivatives \((\varepsilon_x, \kappa, \kappa_z, \ldots, \theta', \theta'')\) are also displayed in Fig. 1. In addition, each individual element cut out of the cross section is also obtained with 3 forces in the local plane \((n_x, t, n_q)\) and the corresponding local strains \((\varepsilon_x, \gamma, \varepsilon_q)\). All these unknown strains and distortion quantities are used directly in the calculation as degrees of freedom.

Using the equilibrium relationship on the element level

\[
\frac{\partial t}{\partial q} + \frac{\partial n_l}{\partial l} = 0
\]

\[
\frac{\partial n_q}{\partial q} + \frac{\partial t}{\partial l} = 0
\]

expressions for the local forces \(t(s)\) and \(n_q(s)\) are derived by substitution and partial differentiation.

\[
t(s) = - \int_{s=0}^{s} \frac{\partial n_l}{\partial l} dq + C_0
\]

\[
n_q(s) = \int_{s=0}^{s} \left( \int_{s=0}^{s} \frac{\partial^2 n_l}{\partial l^2} dq + C_1 \right) dq + C_2
\]

These expressions allow the evaluation of the forces acting on the q-face using forces on the l-face of the cross section. For width cross sections, additional forces perpendicular to the middle line in the p direction need to be taken into account.

The global equilibrium relationships of the beam theory on the whole cross section are

\[
\begin{align*}
\frac{\partial N}{\partial x} &= -p_x, & \frac{\partial V_y}{\partial x} &= -p_y, & \frac{\partial V_z}{\partial x} &= -p_z \\
\frac{\partial T_y}{\partial x} &= -m_x, & \frac{\partial M_y}{\partial x} &= V_z, & \frac{\partial M_z}{\partial x} &= -V_y \\
\frac{\partial^2 N}{\partial x^2} &= 0, & \frac{\partial^2 M_x}{\partial x^2} &= -p_z, & \frac{\partial^2 M_y}{\partial x^2} &= p_y
\end{align*}
\]
From these relations the shear forces \( V_z \) and \( V_y \) between the starting points of the cross section \( s = 0 \) and the present location \( s \) can be calculated, see Hartung, Krebs [3].

\[
\begin{align*}
V_z &= -z(s) \cdot N_z(s) + M_y(s) \cdot z(s) \cdot \sum_{k=0}^{z(s)} p_{s,k} + \sum_{k=0}^{z(s)} z_k \cdot p_{s,k} \\
V_y &= -y(s) \cdot N_y(s) - M_z(s) \cdot y(s) \cdot \sum_{k=0}^{y(s)} p_{s,k} + \sum_{k=0}^{y(s)} y_k \cdot p_{s,k}
\end{align*}
\]  

(2.4)

Finally, the shear forces \( t_z(s_m) \) and \( t_y(s_m) \) or the transformed forces \( t(s_m) \) and \( t_{lp}(s_m) \) in the local coordinate system of the element are easily determined.

\[
\begin{align*}
t_z &= \frac{\Delta V_z(s_m)}{\Delta s} \quad \text{with} \quad \Delta V_z(s) = V_z(s) - V_z(s - 1) \\
t_y &= \frac{\Delta V_y(s_m)}{\Delta s} \quad \text{with} \quad \Delta V_y(s) = V_y(s) - V_y(s - 1)
\end{align*}
\]  

(2.5)

The normal force \( n_q(s_m) \) and the shear force \( t_{qp}(s_m) \) on the q-face of the element are calculated in a similar way, see Retzepis, Hartung [2]. Using the above forces, the equilibrium conditions on the global as well as the local level are fulfilled.

The state of strain is described using the strain values \( (\varepsilon_x, \kappa_z, \kappa_z) \) as well as their first and second derivatives \( (\varepsilon_x', \varepsilon_x'', \kappa_z', \kappa_z'', \kappa_z''' \kappa_z'''' \) \). To obtain the effects of torsion, the first and second derivatives of the angle of twist \( \vartheta', \vartheta'' \) have to be introduced. For closed parts of the cross section, e.g. box girders of one or more compartments, section cuts have to be introduced. Continuity at the point of the section cut has to be fulfilled using the method of consistent displacements.

[Fig. 2 Section cut through a closed part of a cross section]

Constitutive equations of the required accuracy are used as will be demonstrated in the verification test. These equations allow the direct evaluation of the stress using the corresponding strain.

\[
\begin{align*}
\sigma_1 &= f_1(\varepsilon_1, \varepsilon_2) \\
\sigma_2 &= f_2(\varepsilon_1, \varepsilon_2)
\end{align*}
\]  

(2.6)

Both stress and strain are expressed in their principal directions and are assumed to be coaxial. Fig. 3 shows the stress-strain diagrams for reinforced steel and concrete which are used for a reinforced concrete cross section. The modelling of the tension stiffening effect (TS) has been carried out in a simple way by modifying the stress-strain diagram of concrete in the tension region.
Points 1 – 4 of the diagram characterize the different states of reinforced concrete behaviour: Point 1 represents the beginning of cracking, point 2 the state of final cracking, point 3 the state of yielding in the reinforcement and finally, point 4 indicates the full opening of the crack. The calculation of these points requires knowledge of the behaviour of both concrete and reinforcement as well as of the effective concrete area around the available reinforcement.

[Fig. 3 Stress – strain diagrams for reinforced steel and concrete]

By using the above equilibrium and geometric conditions in combination with the constitutive laws, a solution becomes possible. For the 11 known forces, 11 unknown strain and distortion values can be calculated. The solution procedure is performed on three levels. The start of the procedure begins on level 3 for each element of the cross section, continuing on level 2 for each closed part of the cross section and finishing on level 1 of the entire cross section.

**Level 1 (cross section as a whole part)**

<table>
<thead>
<tr>
<th>Level</th>
<th>Force Values</th>
<th>Strain or Distortion Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>N, M_y, M_z</td>
<td>ε_x, ε_y, ε_z</td>
</tr>
<tr>
<td>1.2</td>
<td>N' = -p_x, M_y' = V_z, M_z' = -V_y</td>
<td>ε_x', ε_y', ε_z'</td>
</tr>
<tr>
<td>1.3</td>
<td>N'' = 0, M_y'' = -p_y, M_z'' = p_y</td>
<td>ε_x'', ε_y'', ε_z''</td>
</tr>
</tbody>
</table>

**Level 2 (closed part(s) of the cross section)**

<table>
<thead>
<tr>
<th>Force Values</th>
<th>Strain or Distortion Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>T_x, t_0</td>
<td>9', Δu = 0</td>
</tr>
<tr>
<td>T_x' = -m_x, t_0'</td>
<td>9'', Δu' = 0</td>
</tr>
<tr>
<td>v_0y, v_0z, m_0x</td>
<td>Δv = 0, Δw = 0, Δφ = 0</td>
</tr>
</tbody>
</table>

**Level 3 (for each element of the cross section)**

<table>
<thead>
<tr>
<th>Force Values</th>
<th>Strain or Distortion Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>t, n_q</td>
<td>γ, ε_q</td>
</tr>
</tbody>
</table>

The matrix formulation on each of these levels is shown in Retzepis, Hartung [2]. This algorithm has been implemented in a computer program (KuK-QuMe).

3. **Verification**

Many verification tests have been carried out, beginning with analytical and numerical solutions of linear elasticity problems, see Retzepis, Hartung [2]. The main focus, however, has been on the material behaviour of reinforced and prestressed cross sections under arbitrary loading. Any type of prestressing is applicable using bonded and unbonded tendons or a combination of these.

[Fig. 4 Kupfer / Baumann [5] test beam 69/1 (Length in cm)]
The behaviour of a reinforced cross section under bending moments and normal forces is well known and doesn’t involve any difficulties. For this reason, the focus has been on tests involving shear forces or a combination of shear with bending. Many beam tests carried out in Germany have been used for verification. At the University of Munich, beam tests have been carried out to check the effectiveness of different forms of stirrups under high shear forces. Fig. 4 shows the Kupfer / Baumann test beam 69/1 with reinforcement (for area A1 and A2). Concrete C 25/30 ($f_{ck,cube} = 32$ Mpa) and reinforcement BSt 420/500 ($f_{yd} \approx 465$ Mpa) have been applied. The point load F was increased steadily till the SLS. After 50 repetitions of loading and unloading, the load was increased till failure. Fig. 5 shows the cracking of the beam at ultimate load. The typical shear cracking initiated by cracks due to bending can be observed. The upper picture shows the crack pattern observed in the test. In the same picture the calculated directions of the cracks are marked in red. Zone D characterizes the discontinuity area (direct influence of the point load) with an approximate length equal to the height of the beam. Zone B is the continuity area. In the lower picture, the calculated principal strains and their directions are shown. The results of the calculation correspond very well to the test.

[Fig. 5 Crack pattern in test (photo) and calculation (red) for test beam 69/1]

Fig. 6 shows the measured and the calculated stress in the stirrup at position 5 (see area A1 in Fig. 4) over the shear stress. The tension stiffening effect has been considered in the calculation. In the diagram, the value of the stress in the middle part of the stirrup (marked as calculation **) and at the bottom part of the stirrup (marked as calculation *) are given. During the test, the stress in the middle part of the stirrup was measured. Again, a good agreement between the test and the calculation can be observed. Furthermore, the calculation results show the possible variation span of the measured value allowing a realistic estimation of the beam behaviour.

[Fig. 6 Shear stress – stirrup stress diagram for test beam 69/1]

For additional verification of the numerical results using the above theory, two recent beam tests have been carried out at the University of Kaiserslautern in Germany, see Krebs, Schnell, Hartung [4]. The most remarkable feature of these tests was that the numerical calculations were performed first and the experimental tests were done afterwards to prove the numerical results. Two geometrically identical beams were tested, one was prestressed using internal unbonded tendons and the other one was a plain reinforced beam. Fig. 7 shows the prestressed beam (beam 1).

[Fig. 7 Kaiserslautern prestressed test beam, see Krebs, Schnell, Hartung [4]]

The reinforcement of the two beams has been chosen in such a way that collapse due to shear could occur in the marked area. Before beginning the test all the necessary material parameters were determined. Applying the developed theory, the ultimate load as well as the strains have been
evaluated. Fig. 8 shows the calculated primary strains with the estimated collapse load of prestressed beam. In the calculation, the tension stiffening effect (TS) has been considered. According to this calculation, the shear collapse would be initiated by the yielding of the stirrups between 1170 kN and 1260 kN. The first value defines the local start of the yielding of the stirrups (‘local’ in Fig. 8), the second value corresponds to the yielding of the stirrups over their whole length (‘complete’ in Fig. 8). The two additional dotted curves in the Figure demonstrate the influence of the tensile stress of concrete $f_{ct}$, one curve having been evaluated with a constant tensile stress of concrete of $f_{ct} = 1.1 \text{ N/mm}^2$ and the second curve without any tensile stress ($f_{ct} = 0$). In addition, the values according to the German codes are given (708 kN according to the ‘old’ code DIN 4227, Part 1 (1988) and 865 kN according to the current code DIN 1045-1 (2001)). The collapse of the beam in the test occurred at 1256 kN.

**[Fig. 8 Primary strain and load capacity of the prestressed beam]**

Fig. 9 shows the force – strain diagram of the calculated and the measured (DMS 1 & 2) values of strains in the stirrups in beam 1 (prestressed) and beam 2 (reinforced) at the cross section R1 (see Fig. 7).

**[Fig. 9 Strains in the stirrups at the cross section R1 of beam 1 and 2]**

Many other tests have been used for verification which demonstrates the very good agreement between the results of the calculations with those of the tests. Applying this theory, a powerful new tool is available which allows much more realistic results compared to the established classical methods.

### 4. Practical Application

Some of the practical applications of the developed theory are shown below. As mentioned before, every arbitrary concrete cross section and any type of loading can be analyzed. A prestressing by bonded or unbonded tendons is possible. Also, cross sections of segmental structures can be calculated.

Using this tool, interaction diagrams can be calculated accurately. They show the bearing capacity of the cross section under the combined action of forces. Fig. 10 presents the $N-M_z-V_z$ interaction diagram of a T-cross section. Mild steel bars and prestressed bonded steel tendons are used as reinforcement. Any combination of $N-M_z-V_z$ values outside the closed surface of the interaction diagram indicates collapse. The distance of a point ($N-M_z-V_z$ value) to the interaction surface represents the safety factor which is available.
The analysis of the fatigue behaviour of the structure requires the calculation of the stresses in the individual members. This control can easily be performed using moment – stress diagrams. For example, the fatigue behaviour of the anchorage and the coupling joints of tendons in prestressed structure have to be checked; the coupling joints of tendons being the most sensitive points of the structure.

In the past many defects in prestressed bridge structures occurred at the coupling joints of tendons. These defects affected bridges in Germany as well as in several other countries. For rehabilitation purposes, a precise calculation of the existing stress in the prestressed tendons is therefore essential. Fig. 11 shows the development of the stress in the prestressed tendon at the coupling joint of a bridge under increasing loading. The global safety factor \( \gamma_{\text{Global}} \) has been chosen as the ordinate axis. The effect of a reduced prestressing force (70% P) on the stress in the tendon is also shown. In addition, the increase of stress in the tendon due to the shear force \( V_z \) and the torsion moment \( T_z \) can be seen.

The theory has already been put into practice. One of the first applications was a bridge with a fault in the web of the box girder during concreting (Fig. 12). A rehabilitation of these defects using synthetic resins was proposed, thus guaranteeing good corrosion protection of the reinforcement. However, it has to be taken into account that the elasticity modulus of synthetic resins is much lower compared to that of the concrete. So this moderate rehabilitation is only possible, if the SLS and the ULS can be proved for the reduced web width of 55 cm; the ULS being in this case the critical state. Applying the standard design method according to the German code DIN 4227, Part 1 (1988), which was valid at the time of the bridge construction, a safety deficit was obtained (global safety factor 1.43 < 1.75). Although the bending with longitudinal force could be satisfied, the capacity of the cross section for shear as well as for shear plus torsion was not sufficient. Also, the calculation done according to the current German code DIN-Fachbericht 102 (2003), which is based on the Eurocode No. 2, showed a deficit of about 10%. By applying the developed theory, however, the ULS with a global safety factor of 1.77 (> 1.75) could be fulfilled. For security reasons the calculations were performed without consideration of the tension stiffening effect and without any tension stress for concrete \( f_{\text{ct}} = 0 \text{ N/mm}^2 \). Based on these results, a moderate rehabilitation of the
points of defect was feasible thus avoiding a costly strengthening of the structure, see Krebs, Schnell, Hartung [4].

In the meantime, many other bridges have been analyzed using the presented theory. In cases of existing bridges the theory helped to find the most efficient and suitable rehabilitation or strengthening method. In fact, it is a matter of saving resources and budgeting for the given structure. Only in the fewest cases does a replacement of the structure became unavoidable.

5. Conclusion

An extension of the classical bending theory has been presented which combines elements of the theory of plane stress. The necessary equilibrium and geometric conditions have been demonstrated. Relatively simple constitutive laws involving only standard material parameters have been used, leading to sufficiently accurate results.

Many tests have been used for verification. The verification focused mainly on high shear stress because the classical bending theory yields poor results in such cases. The comparison of the tests with the calculation shows a very satisfactory agreement. The theory, which has been implemented in a computer program, enables the evaluation of a realistic response of cross sections for the serviceability as well as the ultimate limit state. Any reinforced or prestressed concrete cross section with bonded and/or unbonded tendons under arbitrary loading can be analyzed. Steel and composite cross sections can also be examined.

A wide spectrum of applications can be handled with this theory. It includes interaction diagrams, evaluation of the stresses and strains at different load levels, sensitivity studies of the influence of load variations and the optimization of cross sections by analysis of alternatives. Furthermore, the behaviour of existing cross sections under increased loading can be investigated and effective schemes for rehabilitation and strengthening of existing structures can be developed.

6. References


Extended abstract

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